JOURNAL OF GUIDANCE, CONTROL, AND DYNAMICS
Vol. 15, No. 6, November-December 1992

Untruncated Satellite Perturbations in a Nonrotating Gravitational Field

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In previous work it was assumed that the gravitational field was axisymmetric and due only to interior mass. By use of a particular coordinate system, an orbital theory was developed that led to compact perturbation formulas associated with the general zonal harmonic J_l. No eccentricity truncation was involved in these formulas, although there was an implicit restriction to elliptic orbits. The theory has now been given a logical completeness by three distinct extensions: 1) By allowing for tesseral harmonics, J_{lm}, the assumption of axial symmetry can be dropped; 2) by permitting the index l to be negative, exterior mass can be allowed for; and 3) the formulas can be expressed so that hyperbolic orbits are covered. The extensions are straightforward if the field is time invariant (i.e., nonrotating) and this is assumed. The second extension has involved the most new work, since the planetary equations have to be formulated with a different integration variable when $l < 0$.

Introduction

In recent work[1] the author has developed a theory for satellite motion about an axisymmetric gravitational attractor, giving complete first-order formulas for the perturbations due to the general zonal harmonic J_l. The theory has two complementary components. The first consists of the formulas for the rates of change of the mean elements $\bar{a}$, $\bar{e}$, $\bar{I}$, $\bar{\Omega}$, $\bar{\omega}$, and $\bar{M}$, where the generic mean element (say, $\bar{\zeta}$) is conceptually defined by the removal of the short-period perturbation ($\delta\zeta$) from the osculating element (ζ); thus, its rate of change covers long-period, as well as secular, variation. The second component of the theory combines the short-period perturbations for the six elements into perturbations (δr, δb, δw) in a particular system[2] of spherical polar coordinates, such that b and w are quasilatitude and quasilongitude relative to the mean orbital plane defined by $\bar{I}$ and $\bar{\Omega}$. The coordinates of the "mean satellite" are specified by $\bar{r}$, $\bar{b}(=0)$ and $\bar{w}$, where $\bar{r}$ and $\bar{w}$ are given, in terms of $\bar{a}$, $\bar{e}$, $\bar{\omega}$, and $\bar{M}$, by the usual Keplerian formulas, and the true satellite's position is given by $r = \bar{r} + \delta r$, $b = \delta b$, and $w = \bar{w} + \delta w$. Similarly, $\dot{r} = \dot{\bar{r}} + \delta\dot{r}$, etc.

In the derivation of position and velocity, osculating elements do not have to be computed, since formulas for δr, etc., are so much more compact than the formulas for the six $\delta\zeta$. The *mean* elements are at the heart of the theory, however; thus, their definitions are important. In principle, for any J_l, each $\bar{\zeta}$ is arbitrary to a constant (proportional to J_l), that reflects the intrinsic arbitrariness in the split of ζ into $\bar{\zeta} + \delta\zeta$. But the goal of the simplest possible formulas for δr, δb, and δw leads to an optimum choice for the constant part of each of the six $\delta\zeta$, and the general formulas associated with this choice are given in Ref 1. These formulas, like the much more complicated formulas for the genuinely (nonconstant) short-period parts of the $\delta\zeta$, are irrelevant, however, since the $\bar{\zeta}$ are really defined as the set of mean elements that, via the two components of the theory's algorithm, lead to the correct position and velocity. As a corollary, the algorithm can be inverted (to generate the instantaneous values of the mean elements from a given position and velocity) by an iterative procedure that has been described elsewhere.[3]

The theory was developed[1] with the aid of certain two-index families of inclination and eccentricity functions, and by transforming the integration variable in Lagrange's planetary equations, in their usual form,[4] from t to $\bar{v}$ (mean true anomaly) the equations could be integrated without any truncation. It is at this point that the formulas are separated into the two components, with the integration constants being associated with the $\delta\zeta$ in such a way as to eliminate unnecessary terms in δr, δb, and δw. This elimination is complete for δr and δb, in that all of the terms of these perturbations are covered by general formulas, but the resulting constants that are associated with δe and δM then induce special terms in δw, formulas for which must be appended to the general formulas.

The restriction of the theory to the zonal harmonics was an obvious limitation, since in the usual interpretation for the geopotential the tesseral harmonics, J_{lm}, are of the same order of magnitude as the zonal harmonics (other than J_2). Except for a particular (and serious) complication, however, the generalization to the tesseral harmonics is remarkably straightforward, requiring only that the two-index family of inclination functions be generalized to a three-index family. The complication is associated with the rotation of the field relative to a Newtonian coordinate frame, so that the potential can no longer be regarded as time independent, except within a frame that rotates with the field. There is no difficulty if the integration variable remains t, instead of being changed to $\bar{v}$, as long as a drastic eccentricity truncation is acceptable, and the resulting general formulas were given by the author in earlier papers.[3,4] In the present paper we follow a different philosophy and make the somewhat academic assumption that the field does not rotate. The motivation for this assumption is twofold: 1) A straightforward extension of the untruncated theory is demonstrated, making it that much more logically complete; and 2) the possibility exists that a new theory can be developed that combines the merits of the two existing ones. This philosophy underlies the first of the three extensions that form the subject matter of the paper.

A second limitation of the theory has been to a gravitational field in which the attracting mass is entirely within the orbital region. This field is characterized by harmonic coefficients J_l (or J_{lm}) with positive l. External mass can be allowed for by the inclusion of harmonics with *negative l*, however,[5] and this leads to the second extension. To obtain untruncated formulas, we must essentially restart the analysis, since the change of

Received July 11, 1990; presented as Paper 90-2884 at the AIAA/AAS Astrodynamics Conference, Portland, OR, Aug. 20-22, 1990; revision received Aug. 25, 1991; accepted for publication Oct. 21, 1991.

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variable now has to be to E (eccentric anomaly), not v. The general principles, particularly the combining of the short-period perturbations to give δr, δb, and δw, are unchanged, but an important complication arises. The quantity v is a real geometric angle of physical significance and as such figures in the initial elaboration of the expression for the potential and in the transformation formulas for δr, δb, and δw. This is not true for E, with the corollary that we can no longer conduct the analysis via a family of two-index eccentricity functions but instead require a three-index family. We can use the same inclination functions, however: two-index in an analysis confined to the J_l and three-index when axial symmetry is no longer assumed. Since no additional complications arise when the first extension of the paper is combined with the second, we restrict the new extension to the J_l.

In the original theory and both the extensions referred to earlier, the formulas associated with the arbitrary J_l or J_{lm} are correct (to first order) for elliptic orbits of eccentricity no matter how close to unity. Furthermore, the coordinate formulation for the short-period perturbations is independent of the assumption that an ellipse (rather than a hyperbola) is involved. As a corollary, it has been an easy matter to modify the theory, and its implementing software, so that hyperbolas are covered as well as ellipses. This is the last of the three extensions to be described in the paper; the description is brief, because a more detailed account has been presented elsewhere.[6]

An extended version of the present paper, including some tabulation of the three-index inclination and eccentricity functions, is being prepared.[7]

Extension to Tesseral Harmonics

Since the theory developed for the general J_{lm}, with $l > 0$, parallels so closely the narrower theory[1] for J_l, the extension will be described in a relatively abbreviated manner, which also serves to summarize the previous work. The main interest lies in the particular formulation of the inclination functions and their recurrence relations, which is believed to be novel.

We start by expressing the disturbing potential due to J_{lm} in the following form:

$$U_{lm} = (\mu/r)J_{lm}(R/r)^l P_l^m(\sin\beta)\cos m(\lambda - \lambda_{lm}) \tag{1}$$

where (in the usual interpretation for the Earth), R, β, and λ are equatorial radius, geocentric latitude and longitude, and P_l^m is the associated function of Legendre; $J_{lm}\cos m\lambda_{lm}$ and $J_{lm}\sin m\lambda_{lm}$ are identified with the tesseral harmonic coefficients usually represented[8] as C_{lm} and S_{lm}. Then U_{lm} may be expanded into $l+1$ components U_{lm}^k, where k has the same parity as l and $|k| \le l$. (The traditional expansion[8] involves an always-positive index p, where $2p = l - k$, but there are distinct advantages in the more symmetric k usage, as noted elsewhere.[4,9] In particular, there is an application for values of k of opposite parity to l, for which the values of p are no longer integral.) It is convenient to express U_{lm}^k initially in the following form:

$$U_{lm}^k = (\mu/r)J_{lm}(R/r)^l F_{lm}^k(i)\cos[ku' + m(\Omega' - \lambda_{lm})] \tag{2}$$

where $u' = v + \omega'$, $\omega' = \omega - \frac{1}{2}\pi$, and $\Omega' = \Omega - \nu - \frac{1}{2}\pi$, with ν being interpretable as sidereal angle, which we are treating as a fixed quantity. To express subsequent formulas more concisely, we then rewrite Eq. (2) as

$$U_{lm}^k = -(\mu/p)(p/r)^{l+1}A_{lmk}\cos[ku' + m(\Omega' - \lambda_{lm})] \tag{3}$$

where $p = a(1 - e^2)$, and

$$A_{lmk} = -J_{lm}(R/p)^l F_{lm}^k(i) \tag{4}$$

The quantities A_{lmk} constitute the natural generalization of the A_{lk} of the original theory,[1] only being defined (other than for convenience with zero values) when l and k have the same parity. (In Ref. 1 we could restrict k to nonnegative values by including a factor of 2 in A_{lk} for $k > 0$.) We therefore introduce a family of inclination functions, to be denoted by $A_{lm}^k(i)$ as a generalization of the previous $A_l^k(i)$, which will 1) be defined (and not identical to zero) even when l and k have opposite parity, 2) be (in some useful sense) "normalized," and 3) facilitate simple recurrence relations. Now the author's usage of the functions $A_l^k(i)$ in Ref. 1 was unchanged from his original introduction of these functions in 1966 (Ref. 5), and a generalization to $A_{lm}^k(i)$, with a recurrence relation, was published not long afterward.[10] This was an authentic generalization, assuming $k \ge 0$, in the sense that $A_l^k(i)$ was recovered on setting $m = 0$ in $A_{lm}^k(i)$, but it suffered from an inherent defect: the need for two separate families of functions according to whether, with $k \ge 0$, $m \le k$ or $m \ge k$ [the families coalesced consistently for m = k, and negative k could be handled via $A_{lm}^{-k}(\pi - i)$]. The defect was not serious in regard to the generation of the functions for fixed m and k, with increasing values of l (which is particularly relevant to resonance analysis[9]); this generation involved the same recurrence relation[10] for both families, and the normalized nature of the functions stemmed from their values always being unity (identically) for the minimum possible value of l, namely, max (m, k).

However, the present analysis has pointed to situations in which the separation of the $A_{lm}^k(i)$ into two families is undesirable, e.g., in regard to the need for a recurrence relation with k varying. Therefore, a new definition for the $A_{lm}^k(i)$ is now advocated in which they all belong to a single family, with no difficulty when $k < 0$. The functions are now defined whenever $l \ge m$, but when $m = 0$ and $k \ge 0$ we can no longer identify them with the $A_l^k(i)$ of Ref. 1; indeed, we now have

$$A_{l,0}^{\pm l}(i) = \frac{(2l)!}{2^l(l!)^2}(c \mp 1)^l \tag{5}$$

instead of unity, where $c = \cos i$, the general connection between functions with equal and opposite values of k being

$$A_{lm}^k(i) = (-)^{l-m}A_{lm}^{-k}(\pi - i) \tag{6}$$

The recurrence relation for fixed m and k is now

$$(l-1)(l^2 - m^2)A_{lm}^k(i) - (2l-1)[l(l-1)c - mk]A_{l-1,m}^k(i) + l[(l-1)^2 - k^2]A_{l-2,m}^k(i) = 0 \tag{7}$$

and this can be used to compute the $A_{lm}^k(i)$ from the starting formulas:

$$A_{mm}^k(i) = 1 \quad \text{and} \quad A_{m+1,m}^k(i) = (m+1)c - k \tag{8}$$

The first equation of Eq. (8) again amounts to normalization, but the functions will not be computed as efficiently as before when $m \le |k|$, since the computation will involve unwanted functions with $m < l < |k|$. We have the alternative of computing with fixed l and m, however, which is useful in analysis for a particular J_{lm}. The recurrence relation for this is

$$(l-k)(1+c)A_{lm}^{k+1}(i) - 2(m - kc)A_{lm}^k(i) + (l+k)(1-c)A_{lm}^{k-1}(i) = 0 \tag{9}$$

for which a single starting formula is sufficient, namely, either member of the pair of functions

$$A_{lm}^{\pm l}(i) = \binom{2l}{l+m}[\tfrac{1}{2}(c \mp 1)]^{l-m} \tag{10}$$

For all k with $|k| < l$, the $A_{lm}^k(i)$ can be progressively generated using Eqs. (9) and (10). There is also a recurrence relation for fixed l and k, and many other three-term relations in which only one of the three indices is the same for all three terms. All of these relations[7] follow from the properties of the hypergeometric function,[11] since

$$A_{lm}^{k}(i) = 2^{-l+m}\binom{l+k}{l-m}(c-1)^{l-m} \times {}_2F_1\left(m-l, k-l; m+k+1; \frac{c+1}{c-1}\right) \tag{11}$$

As before, we relate the quantities A_{lmk} to the functions $A_{lm}^{k}(i)$ via a formula that involves a pure rational number, now designated α_{lmk}. The formula is

$$A_{lmk} = (-)^{\frac{1}{2}(l+k)-1} J_{lm}(R/p)^l \times \alpha_{lmk}(1+c)^{\frac{1}{2}(m+k)}(1-c)^{\frac{1}{2}(m-k)} A_{lm}^{k}(i) \tag{12}$$

The α_{lmk} are zero when l and k are of opposite parity, and the nonzero values can be computed from the recurrence relations

$$\alpha_{lmk} = \frac{l+m}{l+k}\alpha_{l-1,m,k-1} = \frac{l+m}{l-k}\alpha_{l-1,m,k+1} \tag{13}$$

with starting values

$$\alpha_{0,m,0} = 2^{-m}m! \tag{14}$$

As before, we will also require a set of quantities that are only nonzero when l and k (now rewritten as κ) are of opposite parity. Introducing bold notation $\boldsymbol{A}_{lm\kappa}$, therefore, we have

$$\boldsymbol{A}_{lm\kappa} = (-)^{\frac{1}{2}(l+\kappa-1)} J_{lm}(R/p)^l \times \boldsymbol{\alpha}_{lm\kappa}(1+c)^{\frac{1}{2}(m+\kappa)}(1-c)^{\frac{1}{2}(m-\kappa)} A_{lm}^{\kappa}(i) \tag{15}$$

where

$$\boldsymbol{\alpha}_{lm\kappa} = \frac{l+\kappa+1}{l}\alpha_{l,m,\kappa+1} = \frac{l-\kappa+1}{l}\alpha_{l,m,\kappa-1} \tag{16}$$

Finally, we require a formula for the (partial) derivative of A_{lmk} with respect to i:

$$A'_{lmk} = (-)^{\frac{1}{2}(l+k)-1} J_{lm}(R/p)^l \alpha_{lmk}(1+c)^{\frac{1}{2}(m+k-1)} \times (1-c)^{\frac{1}{2}(m-k-1)}[(kc-m)A_{lm}^{k}(i) + (l-k)(1+c)A_{lm}^{k+1}(i)] \tag{17}$$

With the new inclination-based functions and other quantities, it is straightforward to follow the analysis of the previous paper,[1] inserting the extra suffix m at all appearances of an A or an α. We use the eccentricity functions unchanged, with the defining relation for the B_{lj} still (with the summation here, and in future equations, to be understood as over j)

$$(p/r)^{l-1} = (1 + e\cos v)^{l-1} = \sum B_{lj}\cos jv \tag{18}$$

where the summation effectively runs from $j = -\infty$ to $j = \infty$, since B_{lj} is taken as zero when $|j| \geq l$. We also require B'_{lj}, the derivative with respect to e, and the quantity E_{lj} given by

$$E_{lj} = q^2 B'_{lj} + (2l-1)e B_{lj} \tag{19}$$

where $q = \sqrt{1-e^2}$.

We use the formula

$$\frac{dv}{dt} = nq^{-3}\left(\frac{p}{r}\right)^2 \tag{20}$$

to change the integration variable in Lagrange's planetary equations to v, exactly as before. [Strictly speaking, the change is to $\bar{v}$, and overbars should be added to all quantities, apart from t, in Eq. (20).] Because of the assumption that the field is nonrotating, we still have the energy integral, which can be expressed in terms of the exact constant a'; as before, a' is adopted as the mean semimajor axis ($\bar{a}$). The exact relation between a and a' is (for the full potential U)

$$a' = a/(1 + 2\mu^{-1}aU) \tag{21}$$

To integrate the planetary equations, we first pick out the term (for each $d\zeta/d\bar{v}$ due to U_{lm}^{k}) that is independent of $\bar{v}$. This term, multiplied by $\bar{n}$, defines $\dot{\zeta}$. Since $\bar{a}$ is invariant, we have five formulas to record; if, for ease of notation, we suppress the indices l, m, and k on the left-hand sides, and the overbars on all quantities on the right-hand sides, the formulas may be expressed as

$$\dot{\bar{e}} = -kne^{-1}q^2 A_{lmk}B_{lk}\sin\theta \tag{22}$$

$$\dot{\bar{i}} = n(kc-m)s^{-1}A_{lmk}B_{lk}\sin\theta \tag{23}$$

$$\bar{s}\dot{\bar{\Omega}} = -nA'_{lmk}B_{lk}\cos\theta \tag{24}$$

$$\bar{e}\bar{s}\dot{\bar{\omega}} = n(ecA'_{lmk}B_{lk} - sA_{lmk}E_{lk})\cos\theta \tag{25}$$

and

$$\bar{e}\dot{\bar{M}} = nq^3 A_{lmk}B'_{lk}\cos\theta \tag{26}$$

where $s = \sin i$ and $\theta = k\omega' + m(\Omega' - \lambda_{lm})$. We express Eqs. (24-26) this way, with factors on the left-hand sides of the equations, to reflect the potential infinities that would otherwise be present on the right-hand sides. These infinities arise from the singularities associated with circular and/or equatorial orbits and can be avoided by the use of alternative elements.[2] The expression for $\dot{\bar{M}}$, representing the effect due to U_{lm}^{k}, is, of course, residual to the energy-derived value n', given by $n'^2a'^3 = \mu$. Of Eqs. (22-26), only Eq. (23) is more than a trivial rewrite of the equations given earlier (with A_{lm} replaced by A_{lmk} and the trigonometric argument $k\omega'$ by θ): the additional replacement, for $\dot{\bar{i}}$, of kc by $kc - m$ reflects the fact that the axial component of the angular momentum, characterized by pc^2, is no longer constant.

For the remaining terms (those involving v) in the planetary equations, indefinite integrals can be written at once, and "constants" of integration are chosen in due course. These integrals, which constitute the short-period perturbations $\delta\zeta$, are combined into δr, δb, δw, through the following formulas (with overbars again omitted):

$$\delta r = (r/a)\delta a - (a\cos v)\delta e + (aeq^{-1}\sin v)\delta M \tag{27}$$

$$\delta b = (\cos u')\delta i + (s\sin u')\delta\Omega \tag{28}$$

and

$$\delta w = \delta\omega + c\,\delta\Omega + [q^{-2}\sin v(1 + p/r)]\delta e + q^{-3}(p/r)^2\delta M \tag{29}$$

From Eqs. (27) and (29) we obtain general formulas that we may associate with U_{lm}^{k}. From Eq. (28), on the other hand, we obtain a formula that cannot be similarly associated since it is expressed in terms of κ, not k. This is of no consequence, however, since the total perturbation due to J_{lm} is in all three cases given by summing the contributions for the appropriate values of k or κ. The formulas themselves involve summations, over all values of j that do not lead to zero denominators. The formulas are

$$\delta r = -(l-1)pA_{lmk}\sum\frac{1}{(k+j+1)(k+j-1)}B_{l-1,j}\cos\phi \tag{30}$$

$$\delta b = -lA_{lm\kappa}\sum \frac{1}{(\kappa+j+1)(\kappa+j-1)} B_{lj}\cos\phi \tag{31}$$

and

$$\delta w = A_{lmk}\sum \frac{1}{(k+j+2)(k+j)(k+j-2)} \times \left\{[2(l+1) - k(k+j)]B_{lj} - \frac{6(l-1)}{(k+j+1)(k+j-1)} B_{l-1,j}\right\}\sin\phi \tag{32}$$

where

$$\phi = ku' + jv + m(\Omega' - \lambda_{lm})\ [= \theta + (k+j)v]$$

in Eqs. (30) and (32), but with κ in place of k in Eq. (31).

The constants in δe and δM are assigned so that terms with $j = -k \pm 1$ do not arise in δr; in δi and $\delta\Omega$ they are assigned so that terms with $j = -\kappa \pm 1$ do not arise in δb; and the constant in $\delta\omega$ is assigned so that there is no term with $j = -k$ in δw. This assignment of constants completes the definition of all the mean elements, since the choice of $\bar{a} = a'$ defines the constant in δa. It remains to give the special formulas for δw when $j = -k \pm 1$ or $-k \pm 2$, the four values (besides $j = -k$) for which there is a zero denominator in Eq. (32):

$$\delta w = -\tfrac{1}{3}A_{lmk}[(2l-k+2)B_{l,k-1} + (l-1)B_{l-1,k-1}]\sin\phi \tag{33}$$

$$\delta w = \tfrac{1}{3}A_{lmk}[(2l+k+2)B_{l,k+1} + (l-1)B_{l-1,k+1}]\sin\phi \tag{34}$$

$$\delta w = -(1/48)A_{lmk}[3(l+k+5)B_{l,k-2} - 19(l-1)B_{l-1,k-2}]\sin\phi \tag{35}$$

and

$$\delta w = (1/48)A_{lmk}[3(l-k+5)B_{l,k+2} - 19(l-1)B_{l-1,k+2}]\sin\phi \tag{36}$$

where ϕ is defined with $j = -k+1$, $-k-1$, $-k+2$, and $-k-2$ in Eqs. (33), (34), (35), and (36), respectively.

This completes the analysis for the general J_{lm}. It will now be exemplified by results for $J_{2,2}$, which have been validated by comparisons with ephemerides produced by numerical integration. Writing $J = J_{2,2}(R/p)^2$ for brevity, we require the A_{lmk} and $A_{lm\kappa}$ given by $A_{2,2,2} = -\tfrac{3}{4}J(1+c)^2$, $A_{2,2,1} = -\tfrac{3}{2}Js(1+c)$, $A_{2,2,0} = \tfrac{3}{2}Js^2$, $A_{2,2,-1} = \tfrac{3}{2}Js(1-c)$, and $A_{2,2,-2} = -\tfrac{3}{4}J(1-c)^2$. From Eqs. (22-26), the only nonzero $\dot{\zeta}$ that arise are given by

$$\dot{i} = 3Jns\sin\chi, \qquad \dot{\Omega} = 3Jnc\cos\chi$$

and

$$\dot{\omega} = -\tfrac{3}{2}Jn(2-5s^2)\cos\chi \tag{37}$$

where $\chi = 2(\Omega - \nu - \lambda_{2,2})$. (Note: we use Ω, ω, and u, rather than Ω', ω', and u', to deal with particular harmonics.) Finally, from Eqs. (30-36) the short-period perturbations in coordinates are given by

$$\delta r = \tfrac{1}{4}Jp[(1+c)^2\cos(\chi+2u) - 6s^2\cos\chi + (1-c)^2\cos(\chi-2u)] \tag{38}$$

$$\delta b = \tfrac{1}{2}Jes\{(1+c)[3\sin(\chi+\omega) - \sin(\chi+u+v)] + (1-c)[3\sin(\chi-\omega) - \sin(\chi-u-v)]\} \tag{39}$$

and

$$\delta w = \tfrac{1}{8}J\{(1+c)^2[\sin(\chi+2u) + 4e\sin(\chi+u+\omega)] + 12es^2[\sin(\chi+v) - \sin(\chi-v)] - (1-c)^2[\sin(\chi-2u) + 4e\sin(\chi-u-\omega)]\} \tag{40}$$

It must not be overlooked, both in general and for the three specific formulas given by Eq. (37), that the $\dot{\zeta}$ induce additional terms of a short-periodic nature, since[1] they are obtained by integrating with respect to v, not t. Thus, $\dot{i}$ in Eq. (37) induces an additional δi of $3Js(v - M)\sin\chi$.

Extension to Negative l

A general expression for the gravitational potential can be obtained by solving Laplace's equation in spherical-polar coordinates (r, β, λ); then[5] the same Legendre function, $P_l^m(\sin\beta)$, with $l \ge 0$, arises as in a solution involving r^l as in a solution involving r^{-l-1}. The latter solution is associated with the traditional harmonic J_{lm}. When we extend to J_{lm} with $l < 0$, therefore, it follows that the function $P_\ell^m(\sin\beta)$ is involved, where the new symbol ℓ is introduced for notational convenience and is defined (for $l < 0$) by

$$\ell = -(l+1) \tag{41}$$

Thus the functions $A_{\ell m}^k(i)$ are immediately applicable, and no new analysis is required in this area.

For simplicity (since there is no difficulty in extending to J_{lm} for a nonrotating field, as in the last section), we restrict the analysis to J_l, with the following general formula for U_l:

$$U_l = -(\mu/r)J_l(R/r)^l P_\ell(\sin\beta) \tag{42}$$

This can be decomposed into $\sum U_l^k$, where k has the same parity as ℓ (i.e., opposite to l), with

$$U_l^k = -(\mu/a)(r/a)^\ell A_{lk}\cos ku' \tag{43}$$

We relate r to a now, rather than p, to anticipate the change of variable from t to E (strictly $\bar{E}$) instead of v. Thus, the formula for the quantity A_{lk} is

$$A_{lk} = J_l(R/a)^l \alpha_{\ell k} s^k A_\ell^k(i) \tag{44}$$

and similarly for the quantity $A_{l\kappa}$. (We can restrict k to be nonnegative, as in Ref. 1, with a factor of 2 included in $\alpha_{\ell k}$, but it is better to allow k to be negative and dispense with this factor.)

The formula for the change of variable is

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \frac{na}{r} \tag{45}$$

and the relation between r and a is

$$r/a = 1 - e\cos E \tag{46}$$

However, in transforming the planetary equations derived from Eq. (43), we must attend to the factor $\cos ku'$ as well as the factor in r/a, where u' is not the trivial function of E that it is of v. This leads to the need for more complicated eccentricity-defined quantities than the B_{lj} introduced in Eq. (18). We define the quantities D_{lkj}, therefore, to satisfy (uniquely) the identity

$$(r/a)^{-l}\cos(kv + \theta) = \sum D_{lkj}\cos(jE + \theta) \tag{47}$$

where θ is arbitrary and imposes the uniqueness (by distinguishing D_{lkj} with positive and negative j). The summation may be regarded as running from $-\infty$ to ∞, but when $|k| \le -l$, which we may certainly assume, the D_{lkj} are only nonzero for $|j| \le -l$. When $l = 0$ (a case with which we are not actually concerned), the only nonzero D is given by $D_{0,0,0} = 1$.

In view of Eq. (46), the D_{lkj} (tabulated in Ref. 7) can in principle be evaluated from their definitions by (47) and the well-known formulas

$$(r/a)\cos v = \cos E - e \quad \text{and} \quad (r/a)\sin v = q \sin E \tag{48}$$

From Eq. (46), for example, with $l = -1$ and $k = 0$, $D_{-1,0,0} = 1$ and $D_{-1,0,1} = D_{-1,0,-1} = -\tfrac{1}{2}e$. We need Eq. (48) as well, for $k = \pm 1$, thus obtaining $D_{-1,1,0} = D_{-1,-1,0} = -e$, $D_{-1,1,1} = D_{-1,-1,-1} = \tfrac{1}{2}(1+q)$ and $D_{-1,1,-1} = D_{-1,-1,1} = \tfrac{1}{2}(1-q)$. These nine values exhaust the nonzero possibilities for $l = -1$. We only need to derive values for $k \ge 0$, since

$$D_{lkj} = D_{l,-k,-j} \tag{49}$$

and D_{lkj}, for $j < 0$, may be derived from $D_{l,k,-j}$ by changing q to $-q$. From the resemblance between the binomial expansions of $(1 + e\cos v)^{-l}$ and $(1 - e\cos E)^{-l}$, it also follows that

$$D_{l,0,j} = (-)^j B_{\ell+2,j} \tag{50}$$

in view of Eq. (18). Less obviously, it can be shown that

$$D_{l,k,0} = \left[\binom{-\ell-2}{k} \Big/ \binom{\ell+1}{k}\right] B_{\ell+2,k} = q^{1-2l} B_{lk} \tag{51}$$

values of B_{lk} for negative l being tabulated in the extended version of Ref. 1.

We can express D_{lkj} in terms of the ubiquitous hypergeometric function, but it is perhaps of greater interest to display the consequent relationship to one of the A functions. Thus,

$$D_{lkj} = (-)^{k+j} 2^{-k} \frac{(-l+k)!(-l-k)!}{(-l+j)!(-l-j)!}$$

$$\times\, e^k q^{-l-k} \left(\frac{1+q}{1-q}\right)^{j/2} A^{j}_{-l,k}(\tan^{-1}\sqrt{-1}\, e) \tag{52}$$

Therefore, a corresponding set of recurrence relations can be written, but here we only record the one corresponding to Eq. (9):

$$(j + l - 1)eD_{l,k,j-1} - 2(j - kq)D_{lkj} + (j - l + 1)eD_{l,k,j+1} = 0 \tag{53}$$

The starting formulas, corresponding to Eq. (10), are

$$D_{l,k,\pm l} = 2^l(-e)^{-l-k}(1 \mp q)^k \tag{54}$$

but only one of these is required in practice—the version with the upper signs. But in computing we must avoid the factor $1 - q$ when e is small; hence, the appropriate version of Eq. (54) may actually be written as

$$D_{l,k,l} = D_{l,k,-\ell-1} = 2^l(-e)^{-l+k}(1+q)^{-k} \tag{55}$$

since $(1 + q)(1 - q) = e^2$.

Rates of Change of the Elements

We now proceed in principle, for each osculating element ζ, to write down the expression for $\dot{\zeta}$ due to U_l^k and then use Eq. (45) to transform from $d\zeta/dt$ to $d\zeta/dE$, expressed as a summation over j. As before,[1] we save space by omitting the expressions for the short-period perturbations, $\delta\zeta$, which follow immediately, apart from the constant in the indefinite integrals, from the E-dependent terms of $d\zeta/dE$. From the terms in $d\zeta/dE$ that are independent of E, however, we list the rate of change of the mean element $\bar{\zeta}$. As before we have a quick route to δa, via Eq. (21). Thus,

$$\delta a = a - a' = -2a' A_{lk}(r/a)^\ell \cos ku' \tag{56}$$

This is an exact equation if (despite our normal convention about elements on the right-hand sides of equations) we take a and u' to be osculating. To have an appropriate form for δn $(= n - n')$, to use in the further integration associated with the analysis of M, we apply Eq. (47) directly to Eq. (56), after introducing factors a/r and r/a so that the latter converts $(r/a)^\ell$ to $(r/a)^{-l}$. Thus (with the convention about mean elements now assumed),

$$\delta a = -2a(a/r)A_{lk}\sum D_{lkj}\cos(jE + k\omega') \tag{57}$$

We can also proceed via the general procedure, which starts with the planetary equation for $\dot{a}$. This is proportional to $\partial U/\partial M$ and leads to

$$\frac{da}{dt} = \frac{na^2}{r} q^{-3} A_{lk}\left(\frac{r}{a}\right)^{-l} \frac{p}{r}\,[(k+\ell)e\sin(ku' - v)$$

$$+\, 2k\sin ku' + (k-\ell)e\sin(ku' + v)] \tag{58}$$

Hence, after the change of variable by Eq. (45),

$$\frac{da}{dE} = \tfrac{1}{2}aq^{-3}A_{lk}\sum[(k+\ell)e^2D_{k-2} + 2(2k+\ell)eD_{k-1}$$

$$+\, 2k(2+e^2)D_k + 2(2k-\ell)e\,D_{k+1}$$

$$+\, (k-\ell)e^2D_{k+2}]\sin(jE + k\omega') \tag{59}$$

where the first and third suffix (always l and j) of each D have been suppressed. By use of the appropriate recurrence relations,[7] this can be reduced first to

$$\frac{da}{dE} = aq^{-2}A_{lk}\sum j(eD_{k-1} + 2D_k + eD_{k+1})\sin(jE + k\omega') \tag{60}$$

and then to

$$\frac{da}{dE} = 2aA_{lk}\sum jD_{l+1,k,j}\sin(jE + k\omega') \tag{61}$$

which is consistent with Eq. (56). For $j = 0$, we recover the result that the mean semimajor axis is constant.

For the analysis of both e and i, it is convenient to start with the planetary equation for $\dot{p}$, which is proportional to $\partial U/\partial\omega$. This leads to

$$\frac{dp}{dt} = 2k\frac{na^2}{r}qA_{lk}\left(\frac{r}{a}\right)^{-l}\sin ku' \tag{62}$$

and hence to

$$\frac{dp}{dE} = 2kaqA_{lk}\sum D_{lkj}\sin(jE + k\omega') \tag{63}$$

From the definition of p, de/dE can be obtained from Eqs. (60) and (63). The term in D_{lkj} then has a factor e^{-1}, but this can be eliminated by use of the recurrence relation with k varying[7]; the result is

$$\frac{de}{dE} = \tfrac{1}{2}A_{lk}q^{-1}\sum[(k - l + jq)D_{k-1} + 2keD_k$$

$$+\, (k + l + jq)D_{k+1}]\sin(jE + k\omega') \tag{64}$$

The simplest expression for $\dot{e}$ follows from the formula prior to elimination of e^{-1}; setting $j = 0$ we get

$$\dot{e} = -kne^{-1}qA_{lk}D_{l,k,0}\sin k\omega' \tag{65}$$

This formula resembles Eq. (22), and the resemblance becomes extremely close if we use Eq. (51) to replace $D_{l,k,0}$ by B_{lk} and allow for the fact that $(R/a)^l$, not $(R/p)^l$, appears in Eq. (44).

Formulas for i follow at once from the constancy of pc^2. Thus, from Eq. (63) we get

$$\frac{\mathrm{d}i}{\mathrm{d}E} = kq^{-1}cs^{-1}A_{lk}\sum D_{lkj}\sin(jE + k\omega') \tag{66}$$

and from Eq. (65) we get

$$\dot{i} = knq^{-1}cs^{-1}A_{lk}D_{l,k,0}\sin k\omega' \tag{67}$$

The planetary equation for $\dot{\Omega}$ is proportional to $\partial U/\partial i$. Proceeding as usual, we derive

$$\frac{\mathrm{d}\Omega}{\mathrm{d}t} = -\frac{na}{r}q^{-1}s^{-1}A'_{lk}\left(\frac{r}{a}\right)^{-l}\cos ku' \tag{68}$$

whence

$$\frac{\mathrm{d}\Omega}{\mathrm{d}E} = -q^{-1}s^{-1}A'_{lk}\sum D_{lkj}\cos(jE + k\omega') \tag{69}$$

Therefore,

$$\bar{s}\dot{\bar{\Omega}} = -nq^{-1}A'_{lk}D_{l,k,0}\cos k\omega' \tag{70}$$

The factor $\bar{s}$ has been applied to the left-hand side to make the right-hand side nonsingular.

We come to ω, and if, as before,[1] we define the quasielement ψ such that $\dot{\omega} = \dot{\psi} - c\dot{\Omega}$, then the planetary equation for $\dot{\psi}$ is proportional to $\partial U/\partial e$ and results for ω can be obtained from the simpler results for ψ. From the equation for $\dot{\psi}$ we derive, after using (45) and a recurrence relation,

$$\frac{\mathrm{d}\psi}{\mathrm{d}E} = -\tfrac{1}{2}e^{-1}q^{-1}A_{lk}\sum[(l - k - jq)D_{k-1} + 2leD_k$$

$$+ (l + k + jq)D_{k+1}]\cos(jE + k\omega') \tag{71}$$

To express $\dot{\bar{\psi}}$ we use Eq. (51) to replace each $D_{l,k,0}$ by B_{lk} and then use Eq. (19) to introduce E_{lk}. On combining $\dot{\bar{\psi}}$ with $\dot{\Omega}$, we get

$$\bar{e}\bar{s}\dot{\bar{\omega}} = nq^{-2l}(ecA'_{lk}B_{lk} - sA_{lk}E_{lk})\cos k\omega' \tag{72}$$

We come finally to M, and define the quasielement ρ so that $\dot{\rho} = \dot{\sigma} + q\dot{\psi}$, where σ is the modified mean anomaly at epoch such that $M = \sigma + n't + \int$, with $\int$ a shorthand for $\int(n - n')\,\mathrm{d}t$. Then the planetary equation for $\dot{\rho}$ is proportional to $\partial U/\partial a$. This leads to

$$\frac{\mathrm{d}\rho}{\mathrm{d}E} = 2\ell A_{lk}\sum D_{lkj}\cos(jE + k\omega') \tag{73}$$

from which the equation for $\mathrm{d}\sigma/\mathrm{d}E$ is immediate, using Eq. (71). Also, since $n - n' = -(3n/2a)\delta a$, Eq. (57) gives us

$$\frac{\mathrm{d}\int}{\mathrm{d}E} = 3A_{lk}\sum D_{lkj}\cos(jE + k\omega') \tag{74}$$

It then follows that, residual to the "unperturbed" value n',

$$\frac{\mathrm{d}M}{\mathrm{d}E} = \tfrac{1}{2}e^{-1}A_{lk}\sum[(l - k - jq)D_{k-1} - 2(l-1)eD_k$$

$$+ (l + k + jq)D_{k+1}]\cos(jE + k\omega') \tag{75}$$

For $\dot{M}$ (minus n'), we get, as in Eq. (26),

$$\bar{e}\dot{\bar{M}} = nq^{3-2l}A_{lk}B'_{lk}\cos k\omega' \tag{76}$$

Perturbations in *r*, *b*, and *w*

On integrating Eqs. (59), (64), (66), (69), (71), and (75), we have the short-period perturbations in the six orbital elements. To get δr, δb, δw, we can apply Eqs. (27–29) as they stand or else, to avoid mixed expressions in v and E, use the following transformed versions:

$$\delta r = (r/a)\delta a + (a^2/r)[(e - \cos E)\delta e + (e\sin E)\delta M] \tag{77}$$

$$\delta b = (a/r)\{[\cos\omega'(\cos E - e) - q\sin\omega'\sin E]\delta i$$

$$+ [\sin\omega'(\cos E - e) + q\cos\omega'\sin E]s\delta\Omega\} \tag{78}$$

and

$$\delta w = q^{-1}\delta L + \tfrac{1}{2}(a/r)^2q^{-1}\{[2(2 - e^2)\sin E - e\sin 2E]\delta e$$

$$- e[3e - 4\cos E + e\cos 2E]\delta M\} \tag{79}$$

where L is the quasielement such that $\dot{L} = \dot{M} + q\dot{\psi}$. We could equally well eliminate v by applying Eq. (48) to the mixed expressions. The general formulas will be simpler in terms of mixed expressions because of the relative simplicity of Eqs. (27–29), but shorter formulas for a particular J_l materialize in the pure-E form, since the number of terms required then reduces to about one-third for δr, one-half for δb, and one-fifth for δw.

But the simplest formulas of all are obtained by eliminating E in favor of v, rather than vice versa. This can be done via the formula reciprocal to Eq. (47):

$$\left(\frac{p}{r}\right)^{-l}\cos(kE + \theta) = \sum_j D_{\bar{l}kj}\cos(jv + \theta) \tag{80}$$

where $D_{\bar{l}kj}(e) = D_{lkj}(-e)$; thus, in pure-$v$ form, all formulas contain an overall factor that is a positive power of r/p. The great merit of pure-v formulas is that odd powers of q disappear, so that coefficients can be expressed (apart from the factor in r/p) as polynomials in e, just as when $l > 0$. General formulas in v alone have not been obtained, but the analysis in this paper concludes with the particular formulas for $l = -2$.

We start with the analysis for δb. This is the simplest of the three perturbations to deal with, although there is the usual complication associated with the occurrence of ω' in Eq. (78) and the A'_{lk}, instead of A_{lk}, in Eq. (69). As before, this leads to a reorganization of the terms associated with different values of k and a consequent regrouping for the possible values of κ, an index that has the opposite parity to k.

From the integrals of Eqs. (66) and (69) we at once get, via Eq. (28),

$$\delta b = -q^{-1}\sum j^{-1}D_{lkj}[kcs^{-1}A_{lk}\cos u'\cos(jE + k\omega')$$

$$+ A'_{lk}\sin u'\sin(jE + k\omega')] \tag{81}$$

Now $kcs^{-1}A_{lk} + A'_{lk}$ can be expressed[1] in terms of $A_\ell^{k-1}(i)$ and $kcs^{-1}A_{lk} - A'_{lk}$ in terms of $A_\ell^{k+1}(i)$, and these quantities arise from Eq. (81) as multiples of $\cos(jE + k\omega' - u')$ and $\cos(jE + k\omega' + u')$, respectively. Instead of taking this pair of contributions as applying for the same k, therefore, and contributing to δb_{lk} (the summation over j is irrelevant), we set $k = \kappa + 1$ in the first contribution and $k = \kappa - 1$ in the second, the contributions now feeding into δb_{lk}. This leads to

$$\delta b = \tfrac{1}{2}\ell q^{-1}A_{l\kappa}\sum j^{-1}[D_{l,\kappa+1,j}\cos(jE - v + \kappa\omega')$$

$$- D_{l,\kappa-1,j}\cos(jE + v + \kappa\omega')] \tag{82}$$

The summation here is in principle over all nonzero values of j, the exclusion of zero being equivalent to the setting of the constants in δi and $\delta\Omega$ to zero. In practice, however, we can assume $|j| \le \ell + 1$, since only then are the D_{lkj} nonzero; hence, Eq. (82) yields $4(\ell+1)$ terms for each value of κ. For l odd, κ takes $\frac{1}{2}\ell$ possible values, and for l even, it takes $\frac{1}{2}(\ell+1)$, but for $\kappa=0$ the same cosines occur (oppositely paired) for equal and opposite values of j. Thus, the number of terms in the overall formula for δb, for *any* $l \le -1$, is always $2l(l+1)$.

The total number of terms reduces to about one-half if we eliminate v from Eq. (82) to obtain a pure-E formula, the actual number then being $\frac{1}{2}(2l^2+l-1)$ if l is odd and $\frac{1}{2}(2l^2+l)$ if l is even. To collect all the multiples of the same cosine term, we have to organize the j summation in three different ways (for a fixed value of κ), and the final result of the analysis is (with the first D suffix suppressed)

$$\delta b = \frac{1}{4}\ell\frac{a}{qr}A_{l\kappa}\sum\{(j+1)^{-1}[(1+q)D_{\kappa+1,j+1}-(1-q) \times D_{\kappa-1,j+1}] - 2j^{-1}e[D_{\kappa+1,j}-D_{\kappa-1,j}] + (j-1)^{-1} \times [(1-q)D_{\kappa+1,j-1}-(1+q)D_{\kappa-1,j-1}]\}\cos(jE+\kappa\omega') \tag{83}$$

It has been assumed that the constants in δi and $\delta\Omega$ are still zero. By choosing suitable nonzero values we can eliminate the terms in Eq. (83) with $j = \pm 1$, but the coefficient of the term with $j=0$ will then alter.

As already noted, the best general formula would be expressed (free of q) by means of v, not E. The formula has the form

$$(r/p)^{\ell}A_{l\kappa}\sum S_{l\kappa j}\cos(jv+\kappa\omega')$$

but general expressions for the functions of eccentricity, $S_{l\kappa j}$, have not been found.

Turning to δr, we get the formula involving mixed expressions by direct application of Eq. (27). The result may be written with only the second D suffix displayed, as

$$\delta r = -\tfrac{1}{2}aA_{lk}q^{-1}\sum\Big(4qD_k\gamma_j - j^{-1}\{[(k-l+jq)D_{k-1} + e(k+l-1)D_k]\gamma_{j+} + [(k+l+jq)D_{k+1} + e(k-l+1)D_k]\gamma_{j-}\}\Big) \tag{84}$$

where $\gamma_j = \cos(jE+k\omega')$, and for γ_{j+} and γ_{j-} the argument also includes $+v$ and $-v$, respectively. The number of terms that arise overall (for all relevant k) is then $\frac{1}{2}(6l^2-l+1)$ for l odd and $\frac{1}{2}(6l^2-l)$ for l even. These numbers reflect the fact that, with zero constants in δe and δM (but not δa), the terms in γ_{j+} and γ_{j-} in Eq. (84) are absent when $j=0$.

To obtain the general pure-E formula, we can apply Eq. (48) to Eq. (84), or, alternatively, start again using Eq. (77). Details of the analysis[7] are too lengthy to include here, but an important simplification of the results comes from the use of the formula

$$D_{lkj} = -\tfrac{1}{2}(eD_{l+1,k,j+1} - 2D_{l+1,k,j} + eD_{l+1,k,j-1}) \tag{85}$$

Then the desired formula may be written as

$$\delta r = -\frac{1}{4}\frac{a^2}{r}A_{lk}\sum[(j+1)^{-1}R_{j+} + j^{-1}R_j + (j-1)^{-1}R_{j-} + R_{ja}]\cos(jE+k\omega') \tag{86}$$

where the formulas for R_{j+}, R_j, R_{j-}, and R_{ja} are given in terms of the $D_{l+1,k,j}$ in Ref. 7. [The R_{ja} component represents the effect of δa; it can be derived from Eq. (57), via Eq. (85) and redefinition of j.] By using Eq. (86) instead of Eq. (84), we bring the number of terms required down to $\frac{1}{2}(2l^2-3l+1)$ for l odd and $\frac{1}{2}(2l^2-3l)$ for l even. It has been assumed that the constants δe and δM are still zero, but by appropriate choice of nonzero values we can eliminate the terms in Eq. (86) with $j=\pm 1$; however, the coefficient of the term with $j=0$ will then alter.

As with δb, the best general formula would be expressed by means of v. It is of the form

$$a(r/p)^{\ell+1}A_{lk}\sum R_{lkj}\cos(jv+k\omega')$$

for eccentricity functions, R_{lkj}, not yet analyzed.

Turning finally to δw, it will be clear from the implicit presence, in Eq. (29), of $\sin 2v$ and $\cos 2v$ in the coefficients of δe and δM, respectively, that we require σ_{j++} and σ_{j--}, as well as σ_{j+} and σ_{j-}, in the "mixed formula"; all of these quantities are defined in analogy with γ_{j+} and γ_{j-}, with σ_j being $\sin(jE+k\omega')$. Thus, the formula is

$$\delta w = \frac{1}{4}A_{lk}q^{-3}\sum j^{-1}\Big\{[(l-k-jq)D_{k-1} - e(l-1+k)D_k](e\sigma_{j++}+4\sigma_{j+}) + [3e(l-k-jq)D_{k-1} - 2(4l-2-e^2l-e^2)D_k + 3e(l+k+jq)D_{k+1}]\sigma_j + [(l+k+jq)D_{k+1} - e(l-1-k)D_k](4\sigma_{j-}+e\sigma_{j--})\Big\} \tag{87}$$

The number of terms, for a given J_l, is then $5l^2$, regardless of whether l is odd or even; this number reflects the absence of all terms when $j=0$, the constants in δe and δM being assumed to be zero.

To obtain the pure-E formula for δw, we apply Eq. (48) to Eq. (87), or, alternatively, start again from Eq. (79). The resulting formula, presented in the same way as for δr, is

$$\delta w = \frac{1}{4}\left(\frac{a}{r}\right)^2 A_{lk}\sum[(j+2)^{-1}W_{j++} + (j+1)^{-1}W_{j+} + j^{-1} \times W_j + (j-1)^{-1}W_{j-} + (j-2)^{-1}W_{j--}]\sin(jE+k\omega') \tag{88}$$

where the formulas for W_{j++}, etc., are given in Ref. 7. By using Eq. (88), instead of Eq. (87), we reduce the number of terms to $\frac{1}{2}(2l^2-3l-1)$ for l odd and $\frac{1}{2}(2l^2-3l)$ for l even.

As with δr and δb, the best general formula would be expressed by means of v. This formula is of the form

$$(r/p)^{\ell}q^{-2}A_{lk}\sum T_{lkj}\sin(jv+k\omega')$$

for eccentricity functions, T_{lkj}, not yet analyzed.

Results for $l = -2$

For $l=-2$, we have $\ell = 1$. There is just one nonnegative value of k, (viz., 1) and one for κ (viz., 0). For brevity, we write $J = J_{-2}(a/R)^2$, $\bar{A} = A_{-2,1} = Js$, and $\underline{A} = A_{-2,0} = -Jc$. Then the $\dot{\bar{\zeta}}$ are given by

$$(\dot{\bar{e}}, \dot{\bar{i}}, \bar{s}\dot{\bar{\Omega}}, \bar{e}\bar{s}\dot{\bar{\omega}}, \bar{e}\dot{\bar{M}}) = \tfrac{3}{2}Jnq^{-1}[-q^2s\cos\omega, ec\cos\omega, ec\sin\omega, (s^2-e^2)\sin\omega, -q(1+4e^2)\sin\omega] \tag{89}$$

and each $\dot{\bar{\zeta}}$ induces a short-period contribution of $(\dot{\bar{\zeta}}/\bar{n})(\bar{E}-\bar{M})$, additional to the $\delta\zeta$ taken care of by δr, δb, and δw.

Because of the simplicity of Eq. (82) when $\kappa = 0$, the formula for δb is compact, even with mixed terms:

$$\delta b = \tfrac{1}{8} A q^{-1}[e(1-q)\cos(2E+v) - 4(1-q)(2+q) \times \cos(E+v) + 4(1+q)(2-q)\cos(-E+v) - e(1+q)\cos(-2E+v)] \tag{90}$$

The pure-E formula is

$$\delta b = \tfrac{1}{4} A (a/r)[2(2+e^2) - 5e\cos E - e^2 \cos 2E] \tag{91}$$

and the pure-v formula is

$$\delta b = \tfrac{1}{8} A (r/p)[(8+e^2) + 6e\cos v - 3e^2 \cos 2v] \tag{92}$$

Zero constants for δi and $\delta\Omega$ are assumed in all three equations, but a suitable nonzero choice reduces the number of terms to two in either Eq. (91) or Eq. (92).

The mixed formula for δr, as given by Eq. (84), is lengthy, even though we only need $k = 1$. There are 13 terms, multiples of $\sin(jE \pm v + \omega)$ for $0 < |j| \le 2$ and $\sin(jE + \omega)$ for $|j| \le 2$; thus, the formula will not be recorded [but note the change from the cosines, implicit in Eq. (84) via the various γ, to sines, consequent on the transition from ω' in general formulas to ω in particular formulas]. The pure-E formula is

$$\delta r = \tfrac{1}{16} A (a^2/r)\{(1+q)[e^2 S_3 - 2e(8-3q)S_2 - (65 - 58q + 7q^2)S_1] + 16e(10 - 3q^2)S_0 - (1-q) \times [(65 + 58q + 7q^2)S_{-1} + 2e(8+3q)S_{-2} - e^2 S_{-3}]\} \tag{93}$$

where $S_j = \sin(jE + \omega)$. This formula assumes that the constants in δe and δM are zero, although we could choose them to eliminate the terms in $S_{\pm 1}$. The pure-v formula, which can be derived from Eq. (93) via Eq. (80), is

$$\delta r = -\tfrac{1}{16} A a (r/p)^2 \{3e^2(5 + 2e^2)S_3 + 6e(8 + e^2)S_2 + (28 - 18e^2 + 5e^4)S_1 - 4e(14 + e^2)S_0 - e^2(61 - 16e^2)S_{-1} + 6e^3 S_{-2} + 9e^4 S_{-3}\} \tag{94}$$

where now $S_j = \sin(jv + \omega)$, and we could set the constants to eliminate a pair of terms, viz., the terms in $S_{\pm 1}$, $S_{\pm 2}$, or $S_{\pm 3}$.

There are 20 terms in the mixed formula for δw, involving multiples of $\cos(jE + j'v + \omega)$ for $0 < |j| \le 2$ and $|j'| \le 2$; thus, again the formula is not recorded. The pure-E formula is

$$\delta w = \tfrac{1}{16} A (a/r)^2 \{(1+q)[e^2 C_3 + e(3+2q)C_2 - (21 + 52q - 29q^2)C_1] + 2e(17 + 8q^2)C_0 - (1-q) \times [(21 - 52q - 29q^2)C_{-1} - e(3-2q)C_{-2} - e^2 C_{-3}]\} \tag{95}$$

where $C_j = \cos(jE + \omega)$. This assumes zero constants for δe and δM as in Eq. (93), and also now for $\delta\omega$. The pure-v formula is then

$$\delta w = \tfrac{1}{16} A (ar/p^2)\{3e^2(5 + 2e^2)C_3 + 6e(13 + 3e^2)C_2 + (88 + 82e^2 - 5e^4)C_1 + 6e(21 - e^2)C_0 + 3e^2(9 - 4e^2)C_{-1} - 24e^3 C_{-2} - 9e^4 C_{-3}\} \tag{96}$$

where now $C_j = \cos(jv + \omega)$. The choice of nonzero constants, for δe and δM, that eliminate $S_{\pm 3}$ in Eq. (94) would also eliminate $C_{\pm 3}$ in Eq. (96).

Extension to Hyperbolic Orbits

The original theory (with J_2, J_3, and J_4 in the disturbing potential), as implemented by Fortran-77 software, had been tested (with numerically integrated ephemerides as reference "truth") for a range of values of the mean orbital elements, $\bar{e}$ and $\bar{i}$ in particular. Further tests (with $J_{2,2}$ and J_{-2} included) were conducted after the first two extensions that have been described. Details of the testing will be provided elsewhere (see Ref. 3 for the basic philosophy), but the general conclusion has been that the theory is accurate to first order in each harmonic (and to second order in J_2) for the complete range of eccentricity (up to unity) and inclination. The only difficulty arises with near-parabolic orbits when J_{-2} is included in the model (and it would apply to any J_l or J_{lm} with $l < 0$), for a reason that will be discussed later.

Although the theory was developed with only elliptic orbits in mind, the nature of its formulation suggested that an extension to hyperbolic orbits should be very easy. Formulas (30–36), in particular, which give the coordinate perturbations when $l > 0$, were already valid for all e. Furthermore, the introduction of the author's universal elements,[12] where necessary in the software, together with the solution of Kepler's hyperbolic equation,[13] could be seen to involve very little difficulty; the essential factor is that e is effectively replaced by $1 - e$, at crucial junctures, to avoid loss of accuracy when $e \approx 1$. The extension has recently been made and reported[6]; therefore, only the salient points will be discussed here.

There are two points in the original software at which square roots are taken of quantities that are positive for ellipses but negative for hyperbolas. The quantities may be written $N(= \mu/a^3)$ and $Q(= 1 - e^2)$, the square roots (for elliptic orbits) being n and q, respectively. For hyperbolic orbits it turns out best to replace n by $+\sqrt{-N}$ and q by $-\sqrt{-Q}$, since the signs in the formulas such as $\dot{\rho} = \dot{\sigma} + q\dot{\psi}$ [introduced after Eq. (72)] are then preserved. Two other quantities that, as defined for ellipses, become imaginary for hyperbolas are M and E; M is replaced by the hyperbolic mean anomaly, but, less obviously, E is replaced by the negative of the hyperbolic eccentric anomaly as conventionally defined.

The hyperbolic extension has been straightforward only because the structure of the theory (and its implementation) is retained. In particular, this applies to the basic split into the long-term evolution of the orbit, represented by the rates of change of the mean elements, and the short-term behavior, represented by the amalgamation of the short-period element perturbations into coordinate perturbations. The split might appear meaningless for hyperbolic orbits, for which there is no longer a "period," but the difficulty disappears if the short-periodic terms are just defined as those for which the argument has an explicit dependence on v (or else the term is constant). Another example of apparent unnaturalness is associated with the variation of the mean element $\bar{\zeta}$, which arises (for $l > 0$, $\bar{v}$ then being the transformed integration variable) as $\hat{\zeta}\,\bar{v}$. For elliptic orbits this is then expressed as $\hat{\zeta}\bar{n}t + \hat{\zeta}(\bar{v} - \bar{M})$, where $\hat{\zeta}\bar{n}$ is identified with $\dot{\bar{\zeta}}$. For hyperbolic orbits, $\bar{M}$ here is replaced by the (mean) hyperbolic mean anomaly, which it may seem artificial to subtract from $\bar{v}$. Artificiality does not imply invalidity, however, and the minimally modified software works correctly.

The short-period perturbations required no modification at all, since the general formulas for $l > 0$ [Eqs. (30–32)] are already universal. There is not even a problem when $l < 0$, as long as the pure-v formulations of the perturbations are used; thus, Eqs. (92), (94), and (96) already apply to the hyperbola, being free of the quantity q. In connection with the original transformation formulas for the coordinates [Eqs. (27–29)], the apparent difficulty arising from the factors q^{-2} and q^{-3} in Eq. (29) vanishes when we replace these by Q^{-1} and $(Qq)^{-1}$, respectively.

After the necessary software changes had been made, excellent accuracy was found for all values of e, except (when J_{-2} was involved) for values close to unity. Now, if $e = 1$, an orbit is parabolic or rectilinear (or both[12]); we ignore the latter possibility and hence can assume $p \neq 0$. Then, for harmonics with $l > 0$, all perturbation formulas give bounded results, and the software operates consistently for values of e however close to unity; more precisely, it is for values of $1 - \bar{e}$ however close to zero, with exact zero being the sole exception.[6] For harmonics with $l < 0$, however, a fundamental difficulty arises, for both ellipses and hyperbolas, when $e \approx 1$. It results from the use of Eqs. (46) and (47) in inappropriate circumstances, namely, when $e \cos E \approx 1$, so that the sum of terms on right-hand side of Eq. (47) is small but the individual terms are not. This builds potential inaccuracy into the basic split of each ζ into $\bar{\zeta}$ and $\delta\zeta$.

Conclusions

An outline has been given of three extensions to the author's perturbation theory for an axisymmetric gravitational field: 1) to an arbitrary (but nonrotating) field; 2) to a field with zonal harmonics of negative degree; and 3) to hyperbolic orbits. The first extension was essentially trivial. The second involved new analysis, associated with the formulation of the planetary equations in eccentric anomaly instead of true anomaly (it is still possible to express the results in true anomaly, however). The third extension was easy to accomplish since the existing software had only minimal dependence on elliptic assumptions (the fact that eccentric anomaly could be eliminated from formulas derived for the second extension illustrates the point).

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